1. Let $(\Omega, \mathcal{F}, \mu)$ be a Boolean measure space; let μ^* denote the corresponding outer measure. If $\mu^*(E) = 0$, show that E is μ^*-measurable.

   Solution: A subset $E \subseteq \Omega$ is said to be μ^*-measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

   for $A \subseteq \Omega$. Now $(A \cap E) \subseteq E$ and $\mu^*(E) = 0$ given, hence $\mu^*(A \cap E) = 0$. Again $(A \cap E^c) \subseteq A$ and by monotonicity $\mu^*(A) \geq \mu^*(A \cap E^c)$. Therefore

$$\mu^*(A) \geq 0 + \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

   for all $A \subseteq \Omega$. So, we conclude that E is μ^*-measurable.

$\square$

2. Let f be real valued function on a measurable space $(\Omega, \mathcal{B})$. Show that f is measurable iff $E_r \in \mathcal{B}$ for any rational number r, where $E_r = \{\omega \in \Omega : f(\omega) \leq r\}$.

Solution: For r rational number

$$\{\omega \in \Omega : f(\omega) \leq r\} = \bigcap_{n=1}^{\infty} \{\omega \in \Omega : f(\omega) < r + \frac{1}{n}\},$$

countable intersection of measurable sets is measurable. Again

$$\{\omega \in \Omega : f(\omega) < r\} = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : f(\omega) \leq r - \frac{1}{n}\},$$

countable union of measurable sets is also measurable. Hence the proof.

$\square$

3. Let $(\Omega, \mathcal{B}, \mu)$ be a $\sigma-$ finite measure space, and let f be nonnegative integrable function on it. Define $\lambda(E) = \int_E f(\omega) d\mu(\omega)$ for $E \in \mathcal{B}$. Show that λ is a totally finite measure on $(\Omega, \mathcal{B})$.

Solution: We know, μ is a $\sigma-$ finite measure iff there exists a sequence $\{S_n\}_{n=1}^{\infty} \in \mathcal{B}$ s.t. $\cup_{n=1}^{\infty} S_n = \Omega$, $S_1 \subseteq S_2, \cdots$, and $\mu(S_n) < \infty$ for all n.

Now $E \in \mathcal{B} \Rightarrow E \in S_k$ for some k and given that $f \geq 0$ and integrable function, hence

$$\lambda(E) = \int_E f(\omega) d\mu(\omega) \leq \int_{S_k} f(\omega) d\mu(\omega).$$

Since, $\mu(S_k) < \infty$ and E is choosen arbitrarily, hence the result.

$\square$

4. Let f be a real valued measurable function on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$. For $n = 1, 2, \cdots$ define

$$f_n(\omega) = -n, \quad \text{if } f(\omega) < (-n),$$
$$f_n(\omega) = f(\omega), \quad \text{if } |f(\omega)| \leq n,$$
$$f_n(\omega) = n, \quad \text{if } f(\omega) > n.$$

i) Show that f_n is a measurable function for each n.

ii) If f is integrable w.r.t. μ, show that $\int_\Omega f(\omega) d\mu(\omega) = \lim_{n\to\infty} \int_\Omega f_n(\omega) d\mu(\omega)$.

iii) If $\sup_n \int_\Omega |f_n(\omega)| d\mu(\omega) < \infty$, show that f is integrable w.r.t. μ.

Solution: i)

$$f_n = f \cdot \mathbb{1}_{A_n} + (-n) \cdot \mathbb{1}_{B_n} + n \cdot \mathbb{1}_{C_n}.$$

Where

$$A_n = \{\omega : |f(\omega)| \leq n\},$$
$$B_n = \{\omega : f(\omega) < -n\},$$
$$C_n = \{\omega : f(\omega) > n\}.$$

Hence, f_n is measurable function for each n.

ii) f is integrable. Define

$$f_+ := f \cdot \mathbb{1}_{\{\omega : f(\omega) \geq 0\}},$$
$$f_- := -f \cdot \mathbb{1}_{\{\omega : f(\omega) < 0\}},$$
$$f_{n_+} := f_n \cdot \mathbb{1}_{\{\omega : f_n(\omega) \geq 0\}},$$
$$f_{n_-} := -f_n \cdot \mathbb{1}_{\{\omega : f_n(\omega) < 0\}},$$

Then observe $f_+,\ f_-,\ f_{n_+},\ f_{n_-}$ are all measurable. Again,

$$f = f_+ - f_-.$$

Further,

$$0 \leq f_+ \leq |f| \quad \text{a.e.}$$
$$0 \leq f_- \leq |f| \quad \text{a.e.}$$

Therefore, $f_+,\ f_-$ are integrable. Again

$$f_{n_+} \uparrow f_+ \text{ and } f_{n_-} \uparrow f_-.$$

Therefore $|f_n| \uparrow |f|$. Hence, the result follows from MCT.

iii) Since, $|f_n| \uparrow |f|$ as $n \to \infty$ and $\sup_n \int_\Omega |f_n(\omega)| d\mu(\omega) < \infty$ given. Therefore, the result follows from MCT. $\square$